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A Voronovskaya Type Theorem for Poisson–Cauchy Type singular operators

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ABSTRACT

In this article we continue with the study of approximation properties of smooth Poisson–Cauchy Type singular integral operators over the real line. We produce Voronovskaya Type results and give some quantitative results regarding the rate of convergence of these singular integral operators.

1. Introduction

We are motivated by the approximation properties of *Poisson–Cauchy Type singular integral* of a function f defined by the following

$$M_\xi(f;x) := \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\int_{-\infty}^{\infty} f(x+y)\frac{1}{(y^{2\alpha}+\xi^{2\alpha})^\beta}\,dy, \tag{1}$$

for all $x \in \mathbb{R}$, $\xi > 0$, $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, see [11], and see also [5, Chps. 16, 17], [4, Chp. 21], [6]. Other motivations come from [1–3,7–10,12,14].

Next we define the *smooth Poisson–Cauchy Type singular integral operators* $M_{r,n,\xi}(f;x)$ of which basic approximation properties were studied in [11].

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j}j^{-n}, & j = 1,\dots,r, \\ 1-\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^{-n}, & j=0, \end{cases} \tag{2}$$

that is $\sum_{j=0}^{r}\alpha_j = 1$.

Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(n)}$ exists and it is bounded and Lebesgue measurable. We define for $x \in \mathbb{R}$, $\xi > 0$ and $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$ the Lebesgue singular integrals

$$M_{r,n,\xi}(f;x) = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\int_{-\infty}^{\infty}\frac{\sum_{j=0}^{r}\alpha_j f(x+jt)}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt. \tag{3}$$

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Note 1. The operators $M_{r,n,\xi}$ are not in general positive, see [11].

Note 2. In particular we have $M_{1,n,\xi} = M_\xi$.

In Section 2 we give an elementary property of integrals related to (3). Then in Section 3, we will prove Voronovskaya type asymptotic results, see also [13].

2. An auxiliary result

From (3) we also obtain

$$M_{r,n,\xi}(f;x) = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty} f(x+jt)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt. \tag{4}$$

By means of elementary calculations, we obtain

Lemma 3. *For every $\alpha \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\beta > \frac{n+1}{2\alpha}$, and $\xi > 0$, we have*

$$I_n := \int_0^\infty \frac{t^n}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt = \frac{1}{2\alpha\xi^{2\alpha\beta-n-1}}\frac{\Gamma(\frac{n+1}{2\alpha})\Gamma(\beta-\frac{n+1}{2\alpha})}{\Gamma(\beta)}. \tag{5}$$

Proof. See [15, p. 397, formula 595]. □

3. Results

We present our main result.

Theorem 4. *Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(n)}$ exists, $n \in \mathbb{N}$, and is bounded and Lebesgue measurable on $\mathbb{R}$, and let $\xi \to 0+$, $0 < \gamma \leqslant 1$, $\alpha \in \mathbb{N}$, $\beta > \frac{n+1}{2\alpha}$. Then*

$$M_{r,n,\xi}(f;x) - f(x) = \sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor} f^{(2m)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\frac{\Gamma(\frac{2m+1}{2\alpha})\Gamma(\beta-\frac{2m+1}{2\alpha})}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\frac{\xi^{2m}}{(2m)!} + o\big(\xi^{n-\gamma}\big). \tag{6}$$

Proof. We notice by $\frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\int_{-\infty}^{\infty}\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt = 1$, that $M_{r,n,\xi}(c,x) = c$, for any c constant, and therefore we have

$$M_{r,n,\xi}(f;x) - f(x) = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\left(\sum_{j=0}^{r}\alpha_j\int_{-\infty}^{\infty}\big(f(x+jt)-f(x)\big)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right). \tag{7}$$

Using Taylor's formula for f, we have

$$f(x+jt) = \sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k + \frac{f^{(n)}(\theta)}{n!}(jt)^n, \tag{8}$$

with $\theta = \theta(j)$ between x and $x+jt$.

We obtain

$$\begin{aligned} M_{r,n,\xi}(f;x) - f(x) &\overset{(7)}{=} \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})} \\ &\quad\times\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left(\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(x)}{k!}(jt)^k + \frac{f^{(n)}(\theta)}{n!}(jt)^n\right] - f(x)\right)\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right) \\ &= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\left(\sum_{j=1}^{r}\alpha_j\int_{-\infty}^{\infty}\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^k t^k + \frac{f^{(n)}(\theta)}{n!}j^n t^n\right]\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right) \end{aligned}$$

$$= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}j^k\left(\int_{-\infty}^{\infty}\frac{t^k}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right)\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right]\right)$$
$$\overset{(5)}{=} \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\left(\sum_{j=1}^{r}\alpha_j\left[\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}\frac{f^{(2m)}(x)}{(2m)!}j^{2m}\frac{1}{\alpha\xi^{2\alpha\beta-2m-1}}\frac{\Gamma(\frac{2m+1}{2\alpha})\Gamma(\beta-\frac{2m+1}{2\alpha})}{\Gamma(\beta)}\right.\right.$$
$$\left.\left.+\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right]\right)$$
$$= \sum_{j=1}^{r}\alpha_j\left[\left(\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)\frac{\Gamma(\frac{2m+1}{2\alpha})\Gamma(\beta-\frac{2m+1}{2\alpha})}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\frac{j^{2m}\xi^{2m}}{(2m)!}\right)\right.$$
$$\left.+\frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right].$$

Therefore we have obtained

$$M_{r,n,\xi}(f;x)-f(x)-\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\frac{\Gamma(\frac{2m+1}{2\alpha})\Gamma(\beta-\frac{2m+1}{2\alpha})}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\frac{\xi^{2m}}{(2m)!}$$
$$= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\sum_{j=1}^{r}\alpha_j\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt. \tag{9}$$

Hence

$$\Delta_\xi := \frac{1}{\xi^n}\left[\big(M_{r,n,\xi}(f;x)-f(x)\big)-\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor}f^{(2m)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\frac{\Gamma(\frac{2m+1}{2\alpha})\Gamma(\beta-\frac{2m+1}{2\alpha})}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\frac{\xi^{2m}}{(2m)!}\right]$$
$$= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1-n}}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\sum_{j=1}^{r}\alpha_j\frac{j^n}{n!}\int_{-\infty}^{\infty}f^{(n)}(\theta)t^n\frac{1}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt$$
$$= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1-n}}{n!\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}\alpha_j j^n f^{(n)}(\theta)\right)\frac{t^n}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt$$
$$= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1-n}}{n!\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\left[\int_{-\infty}^{\infty}\left(\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^{(n)}(\theta)\right)\frac{t^n}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right]. \tag{10}$$

Call

$$\Phi_n(x,t) := \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f^{(n)}(\theta). \tag{11}$$

Thus

$$\Delta_\xi = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1-n}}{n!\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\left[\int_{-\infty}^{\infty}\Phi_n(x,t)\frac{t^n}{(t^{2\alpha}+\xi^{2\alpha})^\beta}\,dt\right]. \tag{12}$$

Since we assumed that $f^{(n)}$ exists and it is bounded we obtain

$$\|f^{(n)}\|_\infty < M, \quad \text{for some } M \geqslant 0.$$

Therefore

$$\begin{aligned}\left|\Phi_n(x,t)\right| &\leqslant \left(\sum_{j=1}^{r}\binom{r}{j}\right)M\\ &= \left(2^r-1\right)M. \end{aligned}\tag{13}$$

Hence

$$|\Delta_\xi| \leqslant \frac{(2^r-1)M\Gamma(\frac{n+1}{2\alpha})\Gamma(\beta-\frac{n+1}{2\alpha})}{n!\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})} =: \lambda. \tag{14}$$

Consequently we get $(0<\gamma\leqslant 1)$

$$\frac{1}{\xi^{n-\gamma}}\left|\left(M_{r,n,\xi}(f;x)-f(x)\right)-\sum_{m=1}^{\lfloor\frac{n-1}{2}\rfloor} f^{(2m)}(x)\left(\sum_{j=1}^{r}\alpha_j j^{2m}\right)\frac{\Gamma(\frac{2m+1}{2\alpha})\Gamma(\beta-\frac{2m+1}{2\alpha})}{\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}\frac{\xi^{2m}}{(2m)!}\right| \leqslant \lambda\xi^{\gamma}\to 0, \tag{15}$$

as $\xi\to 0+$.

Notice that $n-1-2m\geqslant 0$.

From the last we conclude the claim of the theorem. □

Corollary 5 *($n=1$ case). Let f such that f' exists and is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi\to 0+$, $0<\gamma\leqslant 1$, $\alpha\in\mathbb{N}$, $\beta>\frac{1}{\alpha}$. Then*

$$M_{r,1,\xi}(f;x)-f(x)=o\left(\xi^{1-\gamma}\right). \tag{16}$$

Proof. In Theorem 4 we place $n=1$. □

Corollary 6 *($n=2$ case). Let f such that f'' exists and is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi\to 0+$, $0<\gamma\leqslant 1$, $\alpha\in\mathbb{N}$, $\beta>\frac{3}{2\alpha}$. Then*

$$M_{r,2,\xi}(f;x)-f(x)=o\left(\xi^{2-\gamma}\right). \tag{17}$$

Proof. In Theorem 4 we place $n=2$. □

Corollary 7 *($n=3$ case). Let f such that $f^{(3)}$ exists and is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi\to 0+$, $0<\gamma\leqslant 1$, $\alpha\in\mathbb{N}$, $\beta>\frac{2}{\alpha}$. Then*

$$M_{r,3,\xi}(f;x)-f(x)=\xi^2 f''(x)\left(\sum_{j=1}^{r}\alpha_j j^2\right)\frac{\Gamma(\frac{3}{2\alpha})\Gamma(\beta-\frac{3}{2\alpha})}{2\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}+o\left(\xi^{3-\gamma}\right). \tag{18}$$

Proof. In Theorem 4 we place $n=3$. □

Corollary 8 *($n=4$ case). Let f such that $f^{(4)}$ exists and is bounded and Lebesgue measurable on $\mathbb{R}$. Let $\xi\to 0+$, $0<\gamma\leqslant 1$, $\alpha\in\mathbb{N}$, $\beta>\frac{5}{2\alpha}$. Then*

$$M_{r,4,\xi}(f;x)-f(x)=\xi^2 f''(x)\left(\sum_{j=1}^{r}\alpha_j j^2\right)\frac{\Gamma(\frac{3}{2\alpha})\Gamma(\beta-\frac{3}{2\alpha})}{2\Gamma(\frac{1}{2\alpha})\Gamma(\beta-\frac{1}{2\alpha})}+o\left(\xi^{4-\gamma}\right). \tag{19}$$

Proof. In Theorem 4 we place $n=4$. □

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